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# A contact problem for a piezoelectric actuator on an elasto-plastic obstacle

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## Abstract

A problem of motion of a piezoelectric actuator in contact with an elasto-plastic obstacle is reformulated as a PDE in one spatial dimension with hysteresis in the bulk and on the contact boundary. The model is shown to dissipate energy in agreement with the principles of thermodynamics. The main result includes existence, uniqueness, and continuous data dependence of solutions.

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## 1 Introduction

Piezoelectric or magnetostrictive actuators are often used, thanks to the capability of transforming mechanical energy into the electromagnetic one and vice versa, in micropositioning systems for accurate control of small displacements or in energy harvesting devices. The main technical issue in applications is energy dissipation due to intrinsic hysteresis of the material. Heat released during the process as a result of energy dissipation may in turn have a negative influence on the performance of the system. Reliable and correct estimation of dissipated energy is therefore one of the crucial modeling issues.

It has been observed that in common magnetostrictive materials, such as galferol or terfenol D, hysteresis loops exhibit a self-similar character, that is, at different constant stresses, the magnetization curves have a similar shape with slopes inversely proportional to the stress. A natural idea is thus to model the phenomenon by a single scalar hysteresis operator acting on a self-similar variable. The same self-similarity is observed on the magnetostrictive “butterfly-shaped” curves. It was shown in [4] that if magnetic hysteresis is represented by the Preisach operator and the magnetostrictive hysteresis curve by the associated Preisach hysteresis potential, then the full model with two inputs (magnetic field and stress) and two outputs (magnetization and strain) is in agreement with classical thermodynamics as well as with the engineering intuitive rule that in cyclic processes hysteresis dissipation is proportional to the area of the hysteresis loop.

A thermodynamically consistent theory of hysteresis phenomena in electro-magneto-elastic materials has been proposed in [4], where the self-similar character of the constitutive relations was exploited for the first time with a good agreement with experiments.

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However, small discrepancies have been observed at low fields. The inaccuracy was due to mean field feedback effects which were neglected, so that for example mechanical depolarization or demagnetization was not accounted for. Feedback was theoretically included in a thermodynamically consistent way into the model in [10] in connection with the problem of optimal energy harvesting, and it was recently experimentally confirmed in [3].

This paper is devoted to a mathematical model for the situation that a piezoelectric actuator comes into contact with an elasto-plastic obstacle. This situation typically occurs in applications, where the electromechanical actuator is used for accurate micropositioning of components in computer controlled mass production of complex systems, for example. The actuator here is represented by a 1D rod of length  $L$ , with one end fixed and one end free. The free end can come into contact with an elasto-plastic obstacle. If this happens, then the force acting on this free boundary is activated according to the elasto-plastic characteristics of the obstacle.

The constitutive law for the rod includes elastic, viscous, and piezoelectric effects. The mathematical model consists of a 1D dynamic momentum balance equation combined with the 1D Gauss law for the electrodynamic balance. The length of the rod is so small that the speed of light can be assumed infinite. In this case, the Gauss law implies that the dielectric displacement is only a function of time depending on the boundary condition representing the impressed current, which is assumed to be the given driving force of the process.

Unlike in [1, 5–8, 14, 15], we do not take into account the thermal effects in our model, and we focus on the interaction between piezoelectricity and boundary contact. Coupling the model with heat transfer will be a subject of a subsequent study.

The paper is organized as follows. The modeling issues are discussed in Sect. 2, where we derive the PDE with hysteresis operators in the bulk and on the boundary taking into account the piezoelectric constitutive law and the boundary contact condition. Basic elements of the theory of the Preisach operator are summarized in Sect. 3. Existence, uniqueness, and continuous data dependence results are stated and proved in Sect. 4, and the proofs are based on the Banach fixed point principle with respect to a suitable norm.

## 2 Description of the model

The actuator is represented by a thin piezoelectric bar of length  $L$  which vibrates longitudinally. The state variables are the displacement  $u$  and the electric field  $E$ , the state functions are the stress  $\sigma$  and the dielectric displacement  $D$ . We assume that the bar driven by applied electric current is free to move on the one end as long as it does not hit a material obstacle, while the other end is kept fixed. We consider  $u(x, t)$  to be the  $u_1$  component of the displacement vector at time  $t$  of the material point of spatial coordinate  $x \in (0, L)$ , and  $\sigma$  to be the  $\sigma_{11}$  component of the stress tensor. The motion is governed by the system

$$\rho u_{tt} - \sigma_x = 0, \quad (2.1)$$

$$D_x = 0, \quad (2.2)$$

consisting of the Newton law of motion for the displacement  $u$  with mass density  $\rho$ , and of the Gauss law for the dielectric displacement  $D$ . We denote here for simplicity  $(\cdot)_x \stackrel{\text{def}}{=} \frac{\partial(\cdot)}{\partial x}$

and  $(\cdot)_t \stackrel{\text{def}}{=} \frac{\partial(\cdot)}{\partial t}$ . The stress  $\sigma$  is assumed to satisfy the constitutive equation

$$\sigma \stackrel{\text{def}}{=} A\varepsilon + \nu\varepsilon_t - eE + \sigma_{\text{piezo}} \quad \text{and} \quad \varepsilon \stackrel{\text{def}}{=} u_x, \quad (2.3)$$

where  $\varepsilon$  is the  $\varepsilon_{11}$  component of the strain tensor,  $A > 0$  is constant elasticity modulus,  $e \in \mathbb{R}$  is a constant piezoelectric coupling coefficient,  $\nu > 0$  is a constant viscosity modulus, and  $\sigma_{\text{piezo}}$  is the irreversible hysteretic piezoelectric stress component.

For  $\sigma_{\text{piezo}}$ , we refer to the model of piezoelectricity presented in [10] as a counterpart of the model for magnetostriction proposed in [3, 4]. It consists in assuming that the constitutive relation is given in terms of one auxiliary scalar self-similar variable denoted by  $q$ . We consider a hysteresis operator  $\mathcal{P}$  which admits a potential operator  $\mathcal{U}$  in the sense that the hysteresis energy inequality

$$q \frac{d}{dt} \mathcal{P}[q] - \frac{d}{dt} \mathcal{U}[q] \geq 0 \quad \text{a. e.} \quad (2.4)$$

holds for all absolutely continuous inputs  $q$ . We assume for definiteness that  $\mathcal{P}$  is the Preisach operator as in [9]. Then, in particular, (2.4) holds for Preisach potential operator  $\mathcal{U}$ . We recall the definition of the operators  $\mathcal{P}$  and  $\mathcal{U}$  in the next section.

We assume the piezoelectric constitutive relation in the form

$$\sigma_{\text{piezo}} = f'(\varepsilon)\mathcal{U}[q] + \frac{1}{2}b'(\varepsilon)\mathcal{P}^2[q], \quad (2.5)$$

$$D = e\varepsilon + \kappa E + \mathcal{P}[q], \quad (2.6)$$

$$q = \frac{1}{f(\varepsilon)}(E - b(\varepsilon)\mathcal{P}[q]), \quad (2.7)$$

where  $q$  the self-similar variable,  $f(\varepsilon) > 0$  is a self-similarity function,  $b(\varepsilon)$  is a feedback coefficient, and  $\kappa > 0$  is the dielectric constant. The primes denote derivatives with respect to  $\varepsilon$ . A detailed discussion about physical motivation for self-similarity can be found in [3].

System (2.5)–(2.7) coincides with the model in [9] for  $b(\varepsilon) = 0$ , that is, when the feedback effects are neglected. Recent experimental investigations in [3] carried out in Benevento in the case of magnetostriction illustrate the importance of the feedback term at low fields. This is why we include the feedback term also here similarly as in [10].

Note that the equation for  $q$  in (2.7) is implicit, and we have to prove that it is well posed. To this end, we invoke in Theorem 3.3 a general result on invertibility of the Preisach operator with time-dependent coefficients.

Let us first check that model (2.5)–(2.7) is consistent with classical thermodynamics in the sense that there exists a free energy operator  $\mathcal{F}[\varepsilon, E]$  such that for all absolutely continuous processes we have

$$D_t E + \varepsilon_t \sigma - \frac{d}{dt} \mathcal{F}[\varepsilon, E] \geq 0 \quad \text{a. e.} \quad (2.8)$$

The left-hand side of (2.8) is the total energy dissipation rate which has to be nonnegative in agreement with the first and the second principles of thermodynamics.

We claim that the right choice for the free energy operator is then

$$\mathcal{F}[\varepsilon, E] = \frac{A}{2}\varepsilon^2 + \frac{\kappa}{2}E^2 + f(\varepsilon)\mathcal{U}[q] + \frac{1}{2}b(\varepsilon)\mathcal{P}^2[q] \quad (2.9)$$

with  $q$  as in (2.7). An elementary computation now yields

$$D_t E + \varepsilon_t \sigma - \frac{d}{dt} \mathcal{F}[\varepsilon, E] = f(\varepsilon) \left( q \frac{d}{dt} \mathcal{P}[q] - \frac{d}{dt} \mathcal{U}[q] \right) + \nu \varepsilon_t^2 \geq 0 \quad \text{a. e.} \quad (2.10)$$

by virtue of (2.4), hence (2.8) holds.

For (2.1) we prescribe the Cauchy initial data

$$u(x, 0) = u^0(x) \quad \text{and} \quad u_t(x, 0) = v^0(x) \quad (2.11)$$

for  $x \in (0, L)$ , and boundary conditions at  $x = 0$  and  $x = L$  for  $t \in (0, T)$ , given by

$$u(0, t) = 0 \quad \text{and} \quad \sigma(L, t) = -\mathcal{B}[u(L, \cdot)](t) \quad (2.12)$$

with a boundary contact operator  $\mathcal{B}$  describing the elasto-plastic contact. For details about modeling contact with an elasto-plastic obstacle, we refer to [14]. Here, we just note that an energy inequality analogous to (2.4) has to be satisfied, namely

$$u_t \mathcal{B}[u] - \mathcal{E}[u]_t = |\mathcal{D}[u]_t| \geq 0 \quad \text{a. e.} \quad (2.13)$$

for every absolutely continuous input  $u$ , with potential energy operator  $\mathcal{E}$  and dissipation operator  $\mathcal{D}$ . We only assume the technical assumption on Lipschitz continuity to hold, namely

$$|\mathcal{B}[u_1](t) - \mathcal{B}[u_2](t)| \leq C_B \max_{\tau \in [0, t]} |u_1(\tau) - u_2(\tau)| \quad (2.14)$$

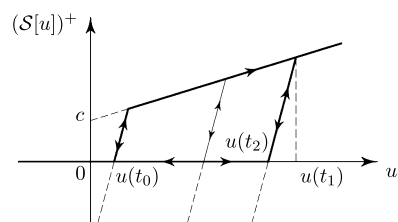
for all  $u_1, u_2 \in C^0([0, T])$  with a constant  $C_B > 0$ .

We may consider the boundary contact operator  $\mathcal{B}$  in the form  $\mathcal{B}[u] = g(\mathcal{S}[u])$ , where  $\mathcal{S}$  is the solution operator  $\mathcal{S} : u \mapsto w = \mathcal{S}[u]$  of the variational inequality

$$\begin{cases} w(t) - au(t) \leq c & \text{for every } t \in [0, T], \\ w(0) = \min\{au(0) + c, bu(0)\}, \\ (bu_t(t) - w_t(t))(w(t) - au(t) - z) \geq 0 \quad \text{a. e. for every } z \leq c, \end{cases} \quad (2.15)$$

with constants  $a > b > 0$ ,  $c > 0$ , where  $a$  is the elasticity modulus of the obstacle,  $b$  is its hardening modulus,  $c$  is its yield point, and  $g$  is a Lipschitz continuous nondecreasing function which vanishes for negative arguments. A typical choice of  $g$  is the positive part  $g(\zeta) = \zeta^+ \stackrel{\text{def}}{=} \max\{\zeta, 0\}$ . To illustrate the meaning of (2.15), assume that the bar touches at some time  $t_0$  the elasto-plastic obstacle for some value  $u(t_0) \geq 0$  of the displacement  $u$ , see Fig. 1, and that  $u$  increases in some time interval  $[t_0, t_1]$ .

The reaction of the obstacle is first elastic with slope  $b$ . It becomes plastic with linear kinematic hardening of slope  $a$  when the yield criterion is reached. If  $u(t)$  starts decreasing after the time  $t_1$ , then  $\mathcal{S}[u](t)$  decreases as well and follows the reversible elastic unloading path with slope  $b$  until the contact is lost at some time  $t_2$ , that is,  $\mathcal{S}[u](t_2) = 0$ . The value  $u(t_2)$  represents a remanent deformation of the obstacle at time  $t_2$ , so that the next contact

**Figure 1** Deformation of an elasto-plastic obstacle

with the obstacle takes place when  $u(t)$  reaches  $u(t_2)$  again. The reader is referred to [14] for some further explanations.

The energy balance (2.13) holds provided we choose

$$\begin{aligned}\mathcal{E}[u] &\stackrel{\text{def}}{=} \frac{1}{b} \left( G(w) + \frac{b-a}{a} G\left( \frac{a}{b-a} (bu - w) \right) \right), \\ \mathcal{D}[u] &\stackrel{\text{def}}{=} \frac{b-a}{ab} \left( G\left( \frac{a}{b-a} (bu - w) + \frac{bc}{b-a} \right) - G\left( \frac{a}{b-a} (bu - w) \right) \right),\end{aligned}$$

where  $G(z) \stackrel{\text{def}}{=} \int_0^z g(s) \, ds$ . Identity (2.13) can easily be checked by a straightforward differentiation, taking into account the fact that  $bu_t - v_t \geq 0$  almost everywhere, and if  $bu_t - v_t > 0$ , then  $w = au + c$ . The Lipschitz continuity (2.14) of this operator is a standard result which goes back to [11], see also [14].

We define the space  $X \stackrel{\text{def}}{=} \{\phi \in W^{1,2}(0, L) : \phi(0) = 0\}$  and state the problem in a variational form as

$$\forall \phi \in X : \int_0^L (\rho u_{tt} \phi + \sigma \phi_x) \, dx + \mathcal{B}[u(L, \cdot)](t) \phi(L) = 0. \quad (2.16)$$

The equation  $D_x = 0$  means that  $D$  is a function of  $t$  only, say,  $D(x, t) = r(t)$ , that is,

$$e\varepsilon + \kappa E + \mathcal{P}[q] = r(t), \quad (2.17)$$

where  $r(t)$  is a function which is known from an additional boundary condition  $D(0, t) = D(L, t) = r(t)$ , corresponding to an impressed (or measured) boundary current.

If no energy supply takes place, that is, if  $r(t) \equiv 0$  in (2.17), then the total energy of the system is nonincreasing in agreement with physical expectation. Indeed, we test (2.16) by  $\phi = u_t$  and use the fact that  $D = 0$ . Then by (2.8) we have

$$\sigma u_{xt} \geq \frac{\partial}{\partial t} \mathcal{F}[u_x, E]. \quad (2.18)$$

We multiply (2.1) by  $u_t$ , and we integrate this expression over  $[0, L]$ . Then, according to (2.11), (2.13), and (2.18), we find

$$\frac{d}{dt} \left( \int_0^L \left( \frac{\rho}{2} u_t^2 + \mathcal{F}[u_x, E] \right) (x, t) \, dx + \mathcal{E}[u](L, t) \right) \leq 0. \quad (2.19)$$

Hence, the sum of kinetic energy, potential energy in the bulk, and boundary potential energy is nonincreasing, which we wanted to check.

Equation (2.17) allows us to eliminate the electric variables from (2.16) similarly as in [9], where the feedback contribution was not taken into account. The strategy is to reformulate the constitutive equation (2.3) in the form

$$\sigma = A\varepsilon + v\varepsilon_t + \mathcal{W}[\varepsilon] \quad (2.20)$$

with a Lipschitz continuous operator  $\mathcal{W} : C^0([0, T]) \rightarrow C^0([0, T])$ . Indeed, by virtue of (2.7), we have

$$E = f(\varepsilon)q + b(\varepsilon)\mathcal{P}[q], \quad (2.21)$$

so that we can rewrite (2.17) as

$$q + \frac{1 + \kappa b(\varepsilon)}{\kappa f(\varepsilon)} \mathcal{P}[q] = \frac{r - e\varepsilon}{\kappa f(\varepsilon)}. \quad (2.22)$$

We need here to represent the auxiliary variable  $q$  in (2.22) as an operator acting on  $\varepsilon$ . In other words, we need to find the inverse operator to the left-hand side of (2.22). In general, the inversion of hysteresis operators is a nontrivial problem. An explicit formula for the inverse Prandtl–Ishlinskii operator as a special case of the Preisach operator has been derived in [12]. The first proof of the existence and Lipschitz continuity of a general inverse Preisach operator given in [2] was based on a geometric idea of evolving memory curves. Here in (2.22), however, the situation is more complicated because of the time dependent factor in front of the Preisach operator, and the memory curve argument fails. The invertibility of such operators was proved much later in [9, Proposition 3.7], and we state the result in Theorem 3.3. Before, we recall some basic elements of the theory of Preisach operators.

### 3 The Preisach operator

We use the following definition of the Preisach operator which is shown in [13] to be equivalent to the original Preisach construction in [16].

**Definition 3.1** Let  $\psi : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function which is Lipschitz continuous in the second variable and such that  $\psi(r, 0) = 0$  for a. e.  $r > 0$ . For a given input  $q \in W^{1,1}(0, T)$ , we define the output  $\mathcal{P}[q] \in W^{1,1}(0, T)$  of a Preisach operator  $\mathcal{P}$  by the integral

$$\mathcal{P}[q](t) \stackrel{\text{def}}{=} \int_0^\infty \psi(r, \xi_r(t)) \, dr \quad \text{for } t \in [0, T], \quad (3.1)$$

where  $\xi_r \in W^{1,1}(0, T)$  is the unique solution of the variational inequality

$$\begin{cases} |q(t) - \xi_r(t)| \leq r & \text{for all } t \in [0, T], \\ \dot{\xi}_r(t)(q(t) - \xi_r(t) - rz) \geq 0 & \text{a. e. and for all } |z| \leq 1, \\ \xi_r(0) = \max\{q(0) - r, \min\{0, q(0) + r\}\}. \end{cases} \quad (3.2)$$

The Preisach operator  $\mathcal{P}$  is called a Prandtl–Ishlinskii operator if  $\psi$  is linear in  $v$ , that is,  $\psi(r, v) = \mu(r)v$  for some  $\mu \in L^1_{\text{loc}}(0, +\infty)$ .

We easily check from (3.2) that  $\xi_r(t) = 0$  for  $r \geq \|q\|$ , where  $\|\cdot\|$  denotes the sup-norm in  $C^0([0, T])$ , so that the integral in (3.1) is meaningful.

The parameter  $r$  is the memory variable, and the mapping  $q \mapsto \xi_r$  introduced in [11] is called the *play operator*. Note that its extension to  $C^0([0, T])$  is Lipschitz continuous with Lipschitz constant 1, that is,

$$|\xi_r^1(t) - \xi_r^2(t)| \leq \max_{\tau \in [0, t]} |q_1(\tau) - q_2(\tau)|. \quad (3.3)$$

For our purposes, it is convenient to reduce the set of admissible functions  $\psi$ , and we adopt the following hypothesis.

**Hypothesis 3.2** *We assume that  $\psi(r, 0) = 0$  for a. e.  $r > 0$ , and*

- (i)  $0 \leq \frac{\partial \psi}{\partial v}(r, v) \leq \mu(r)$  a. e., where  $\mu \in L^1(0, +\infty)$ ,  $\int_0^\infty \mu(r) dr = M$ ;
- (ii) *There exists  $0 < M_1 < +\infty$  such that*

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial v}(r, v) dv dr = M_1; \quad (3.4)$$

- (iii)  $|v| \frac{\partial \psi}{\partial v}(r, v) \leq \mu_2(r)$  a. e., where  $\mu_2 \in L^1(0, +\infty)$ ,  $\int_0^\infty \mu_2(r) dr = M_2$ .

Hypothesis 3.2(i) means that the Preisach operator  $\mathcal{P}$  is dominated by a Prandtl–Ishlinskii operator. Hypothesis 3.2(ii) implies that  $\|\mathcal{P}[q]\| \leq M_1$  for all  $q \in C^0([0, T])$ . It is easy to check that even without Hypothesis 3.2(i), (ii), the Preisach operator satisfies the energy inequality (2.4) with the choice

$$\mathcal{U}[q](t) = \int_0^{+\infty} \Psi(r, \xi_r(t)) dr, \quad (3.5)$$

with

$$\Psi(r, v) = \int_0^v v' \frac{\partial \psi}{\partial v}(r, v') dv'. \quad (3.6)$$

By (3.3), the operator  $\mathcal{P}$  can be extended to  $C^0([0, T])$ , and if Hypothesis 3.2(i) is fulfilled, then the Lipschitz property

$$|\mathcal{P}[q_1](t) - \mathcal{P}[q_2](t)| \leq M \max_{\tau \in [0, t]} |q_1(\tau) - q_2(\tau)| \quad (3.7)$$

holds for all  $q_1, q_2 \in C^0([0, T])$  and all  $t \in [0, T]$ . If moreover Hypothesis 3.2(iii) holds, then the operator  $\mathcal{U}$  is also Lipschitz continuous, and we have

$$|\mathcal{U}[q_1](t) - \mathcal{U}[q_2](t)| \leq M_2 \max_{\tau \in [0, t]} |q_1(\tau) - q_2(\tau)| \quad (3.8)$$

for all  $q_1, q_2 \in C^0([0, T])$  and all  $t \in [0, T]$ .

**Theorem 3.3** *Let Hypotheses 3.2(i), (ii) hold. Then, for every nonnegative function  $b \in C^0([0, T])$  and for every  $w \in C^0([0, T])$ , there exists unique  $q \in C^0([0, T])$  such that*

$$q(t) + b(t)\mathcal{P}[q](t) = w(t) \quad \forall t \in [0, T]. \quad (3.9)$$

Furthermore, let  $b_1, b_2 \in C^0([0, T])$  be such that  $0 \leq b_i(t) \leq \bar{b}$  for all  $t \in [0, T]$ ,  $i = 1, 2$ , and let  $w_1, w_2 \in C^0([0, T])$  be given. Let  $q_1, q_2 \in C^0([0, T])$  be solutions of the equations

$$q_i(t) + b_i(t)\mathcal{P}[q_i](t) = w_i(t) \quad \text{for all } t \in [0, T], i = 1, 2. \quad (3.10)$$

Then we have

$$\|q_1 - q_2\| \leq e^{\bar{b}M} (\|w_1 - w_2\| + M_1 \|b_1 - b_2\|). \quad (3.11)$$

In order to apply Theorem 3.3 to case (2.22), we need to restrict the class of admissible functions  $f(\varepsilon)$  and  $b(\varepsilon)$ . We assume the following conditions to hold.

**Hypothesis 3.4** *The functions  $\varepsilon \mapsto f(\varepsilon)$ ,  $\varepsilon \mapsto b(\varepsilon)$ ,  $\varepsilon \mapsto \varepsilon f(\varepsilon)$ ,  $\varepsilon \mapsto f'(\varepsilon)$ ,  $\varepsilon \mapsto b'(\varepsilon)$  are Lipschitz continuous from  $\mathbb{R}$  to  $\mathbb{R}$ , and there exist constants  $b^\flat < b^\sharp$ ,  $0 < f^\flat < f^\sharp$  such that  $1 + \kappa b^\flat \geq 0$  and the inequalities*

$$f^\flat \leq f(\varepsilon) \leq f^\sharp \quad \text{and} \quad b^\flat \leq b(\varepsilon) \leq b^\sharp$$

hold for all  $\varepsilon \in \mathbb{R}$ .

From Theorem 3.3 and from identities (2.3), (2.5), and (2.21), we obtain the following result.

**Corollary 3.5** *Let Hypotheses 3.2 and 3.4 hold. Then there exists a Lipschitz continuous operator  $\mathcal{W} : C^0([0, T]) \rightarrow C^0([0, T])$  such that the constitutive equation (2.3), (2.5) can be written in the form (2.20).*

#### 4 Existence, uniqueness, and continuous data dependence

The existence and uniqueness result can be stated as follows.

**Theorem 4.1** *Let Hypotheses 3.2, 3.4, and (2.14) hold, let  $r \in C^0([0, T])$  on the right-hand side of (2.17) be given, and let the initial data in (2.11) be given such that  $u^0, v^0 \in X$ . Then system (2.16), (2.20), (2.11) has a unique solution  $u \in C^0([0, T]) \times [0, T]$  such that  $u_{xt} \in L^2((0, L) \times (0, T))$ ,  $u_t \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; X)$ ,  $u_{tt} \in L^2(0, T; X')$ .*

*Proof* Proof For  $v \in C^0([0, T]; X)$  such that  $v_{xt} \in L^2((0, L) \times (0, T))$  and  $v(x, 0) = u^0(x)$ , we find  $u$  with the desired regularity as the solution of the linear problem

$$\begin{aligned} \forall \phi \in X : \\ \rho \int_0^L u_{tt} \phi \, dx + \int_0^L (Au_x + v u_{xt} + \mathcal{W}[v_x]) \phi_x \, dx + \mathcal{B}[v(L, \cdot)] \phi(L) = 0 \quad \text{a. e.} \end{aligned} \quad (4.1)$$

with initial conditions (2.11). We now prove that the mapping  $v \mapsto u$  is a contraction in the space

$$Y \stackrel{\text{def}}{=} \{v \in C^0([0, L] \times [0, T]; X) : v_{xt} \in L^2((0, L) \times (0, T)), v(x, 0) = u^0(x), v(0, t) = 0\},$$



endowed with a suitable norm defined in (4.5). Let  $v, \widehat{v} \in Y$  be given, and let  $u, \widehat{u}$  be the corresponding solutions. We test the difference of Eqs. (4.1) for  $u$  and  $\widehat{u}$

$\forall \phi \in X$ :

$$\begin{aligned} & \int_0^L \rho(u_{tt} - \widehat{u}_{tt}) \phi \, dx + \int_0^L (A(u_x - \widehat{u}_x) + v(u_{xt} - \widehat{u}_{xt}) + \mathcal{W}[v_x] - \mathcal{W}[\widehat{v}_x]) \phi_x \, dx \\ & + (\mathcal{B}[v(L, \cdot)] - \mathcal{B}[\widehat{v}(L, \cdot)]) \phi(L) = 0 \quad \text{a. e.} \end{aligned}$$

Choosing  $\phi = u_t - \widehat{u}_t$  and by using the Lipschitz continuity of  $\mathcal{W}$  and  $\mathcal{B}$ , we get

$$\begin{aligned} & \frac{\rho}{2} \frac{d}{dt} \int_0^L (u_t - \widehat{u}_t)^2 \, dx + \frac{A}{2} \frac{d}{dt} \int_0^L (u_x - \widehat{u}_x)^2 \, dx + v \int_0^L (u_{xt} - \widehat{u}_{xt})^2 \, dx \\ & = - \int_0^L (\mathcal{W}[v_x] - \mathcal{W}[\widehat{v}_x]) (u_{xt} - \widehat{u}_{xt}) \, dx \\ & \quad - (\mathcal{B}[v(L, \cdot)](t) - \mathcal{B}[\widehat{v}(L, \cdot)](t)) (u_t(L, t) - \widehat{u}_t(L, t)) \\ & \leq C \left( \int_0^L |u_{xt} - \widehat{u}_{xt}|^2 \, dx \right)^{1/2} \left( \int_0^L \max_{\tau \in [0, t]} |v_x(x, \tau) - \widehat{v}_x(x, \tau)|^2 \, dx \right)^{1/2} \end{aligned} \quad (4.2)$$

with some constant  $C > 0$ . We have

$$\max_{\tau \in [0, t]} |v_x(x, \tau) - \widehat{v}_x(x, \tau)| \leq \int_0^t |v_{xt}(x, \tau) - \widehat{v}_{xt}(x, \tau)| \, d\tau,$$

so that the right-hand side of (4.2) can be further estimated from above, using the Cauchy–Schwarz and Young inequalities, by the expression

$$\frac{v}{2} \int_0^L (u_{xt} - \widehat{u}_{xt})^2 \, dx + \frac{C^2 t}{2v} \int_0^t \int_0^L (v_{xt} - \widehat{v}_{xt})^2(x, \tau) \, dx \, d\tau.$$

We thus obtain from (4.2) that

$$\begin{aligned} & \frac{d}{dt} \frac{1}{v} \int_0^L (\rho(u_t - \widehat{u}_t)^2 + A(u_x - \widehat{u}_x)^2) \, dx + \int_0^L (u_{xt} - \widehat{u}_{xt})^2 \, dx \\ & \leq \widetilde{C} t \int_0^t \int_0^L (v_{xt} - \widehat{v}_{xt})^2(x, \tau) \, dx \, d\tau \end{aligned} \quad (4.3)$$

with  $\widetilde{C} \stackrel{\text{def}}{=} \frac{C^2}{v^2}$ . This is an inequality of the form

$$\dot{\alpha}(t) + \beta(t) \leq \widetilde{C} t \int_0^t \delta(\tau) \, d\tau \quad (4.4)$$

with nonnegative functions

$$\begin{aligned} \alpha & \stackrel{\text{def}}{=} \frac{1}{v} \int_0^L (\rho(u_t - \widehat{u}_t)^2 + A(u_x - \widehat{u}_x)^2) \, dx, & \beta & \stackrel{\text{def}}{=} \int_0^L (u_{xt} - \widehat{u}_{xt})^2 \, dx \quad \text{and} \\ \delta & \stackrel{\text{def}}{=} \int_0^L (v_{xt} - \widehat{v}_{xt})^2 \, dx. \end{aligned}$$

We multiply (4.4) by  $e^{-\tilde{C}t^2}$  to get

$$\frac{d}{dt} \left( e^{-\tilde{C}t^2} \left( \alpha(t) + \frac{1}{2} \int_0^t \delta(\tau) d\tau \right) \right) + 2\tilde{C}te^{-\tilde{C}t^2} \alpha(t) + e^{-\tilde{C}t^2} \beta(t) \leq \frac{1}{2} e^{-\tilde{C}t^2} \delta(t).$$

We now integrate the above inequality from 0 to  $T$ , and using the facts that  $\alpha(0) = \delta(0) = 0$ ,  $\alpha$  and  $\beta$  are two nonnegative functions, we find

$$\int_0^T e^{-\tilde{C}t^2} \beta(t) dt \leq \frac{1}{2} \int_0^T e^{-\tilde{C}t^2} \delta(t) dt.$$

We conclude that the mapping  $v \mapsto u$  is a contraction in  $Y$  endowed with norm

$$\|v\| \stackrel{\text{def}}{=} \left( \int_0^T \int_0^L e^{-\tilde{C}t^2} |v_{xt}(x, t)|^2 dx dt + \int_0^L |v(x, 0)|^2 dx \right)^{\frac{1}{2}}, \quad (4.5)$$

which implies the existence and uniqueness of solutions.  $\square$

To prove the continuous data dependence, we consider two inputs  $r, \hat{r} \in C^0([0, T])$ , and two sets of initial conditions  $u^0, \hat{u}^0, v^0, \hat{v}^0 \in X$ , and denote the corresponding solutions to (2.16), (2.20), (2.11) by  $u, \hat{u}$ , respectively. We have the following result.

**Theorem 4.2** *Assume that Hypotheses 3.2 and 3.4 hold. Then there exists a constant  $C > 0$  such that*

$$\begin{aligned} & \int_0^T \int_0^L (u_{xt} - \hat{u}_{xt})^2(x, t) dx dt \\ & \leq C \left( \max_{t \in [0, T]} |r(t) - \hat{r}(t)|^2 + \int_0^L |u_x^0 - \hat{u}_x^0|^2 dx + \int_0^L |v^0 - \hat{v}^0|^2 dx \right). \end{aligned}$$

*Proof* Proof Let  $q, \hat{q}$  satisfy (2.22) for  $r, \hat{r}$  and  $\varepsilon = u_x, \hat{\varepsilon} = \hat{u}_x$ , respectively. By using Theorem 3.3 together with Hypothesis 3.4, we have for  $x \in [0, L]$  and  $t \in [0, T]$  that

$$|q(x, t) - \hat{q}(x, t)| \leq C_0 \left( \max_{\tau \in [0, t]} |\varepsilon(x, \tau) - \hat{\varepsilon}(x, \tau)| + \max_{\tau \in [0, t]} |r(\tau) - \hat{r}(\tau)| \right)$$

with a constant  $C_0 > 0$ .

We now construct the operators  $\mathcal{W}$  associated with  $r$  and  $\hat{\mathcal{W}}$  associated with  $\hat{r}$  as in Corollary 3.5 to obtain that

$$|\mathcal{W}[\varepsilon](x, t) - \hat{\mathcal{W}}[\hat{\varepsilon}](x, t)| \leq C_1 \left( \max_{\tau \in [0, t]} |\varepsilon(x, \tau) - \hat{\varepsilon}(x, \tau)| + \max_{\tau \in [0, t]} |r(\tau) - \hat{r}(\tau)| \right)$$

with a constant  $C_1 > 0$ . We now proceed as in the proof of Theorem 4.1 and test the identity

$$\forall \phi \in X :$$

$$\begin{aligned} & \int_0^L \rho(u_{tt} - \hat{u}_{tt}) \phi dx + \int_0^L (A(u_x - \hat{u}_x) + v(u_{xt} - \hat{u}_{xt}) + \mathcal{W}[u_x] - \hat{\mathcal{W}}[\hat{u}_x]) \phi_x dx \\ & + (\mathcal{B}[u(L, \cdot)] - \mathcal{B}[\hat{v}(L, \cdot)]) \phi(L) = 0 \quad \text{a. e.} \end{aligned}$$

Choosing once again  $\phi = u_t - \widehat{u}_t$ , we obtain similarly as in (4.2) that

$$\begin{aligned} & \frac{\rho}{2} \frac{d}{dt} \int_0^L (u_t - \widehat{u}_t)^2 dx + \frac{A}{2} \frac{d}{dt} \int_0^L (u_x - \widehat{u}_x)^2 dx + \nu \int_0^L (u_{xt} - \widehat{u}_{xt})^2 dx \\ &= - \int_0^L (\mathcal{W}[u_x] - \widehat{\mathcal{W}}[\widehat{u}_x])(u_{xt} - \widehat{u}_{xt}) dx \\ & \quad - (\mathcal{B}[u(L, \cdot)](t) - \mathcal{B}[\widehat{u}(L, \cdot)](t))(u_t(L, t) - \widehat{u}_t(L, t)) \\ & \leq C_3 \left( \int_0^L |u_{xt} - \widehat{u}_{xt}|^2 dx \right)^{1/2} \\ & \quad \times \left( \int_0^L \max_{\tau \in [0, t]} |u_x(x, \tau) - \widehat{u}_x(x, \tau)|^2 dx + \max_{\tau \in [0, t]} |r(\tau) - \widehat{r}(\tau)|^2 \right)^{1/2} \end{aligned} \quad (4.6)$$

with a constant  $C_3 > 0$ . Hence, by using the Cauchy–Schwarz and Young inequalities, we find

$$\begin{aligned} & \frac{1}{\nu} \frac{d}{dt} \left( \int_0^L (\rho(u_t - \widehat{u}_t)^2(x, t) + A(u_x - \widehat{u}_x)^2(x, t)) dx \right) + \int_0^L (u_{xt} - \widehat{u}_{xt})^2(x, t) dx \\ & \leq \frac{C_3^2}{\nu^2} t \int_0^t \int_0^L |u_{xt} - \widehat{u}_{xt}|^2(x, \tau) dx d\tau + \frac{C_3^2}{\nu^2} \max_{\tau \in [0, t]} |r(\tau) - \widehat{r}(\tau)|^2, \end{aligned} \quad (4.7)$$

and the assertion is obtained from the Grönwall argument.  $\square$

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##### Authors' contributions

Each author's contribution is 50%. PK did the parts related to piezoelectricity and hysteresis, AP did the parts related to contact mechanics. The final redaction was made in cooperation. Both authors read and approved the final manuscript.

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